LOCAL HEATING OF AN UNBOUNDED ORTHOTROPIC PLATE THROUGH A CIRCULAR AND ANNULAR DOMAIN

V. P. Kozlov, V. S. Adamchik,
and V. N. Lipovtsev

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The regularities of temperature field development in an unbounded orthotropic plate heated through circular and annular domains by an arbitrary heat flux of density $q(r, \tau)$ are established.

1. HEATING THROUGH A CIRCULAR DOMAIN

An unbounded plate (Fig. 1) of height h whose initial temperature is constant and equal to T_0 at all points is heated in the domain of a circle $r < R$ $(z = 0)$ by a heat flux of the specific intensity $q(r, \tau)$. Outside the circle, the plate surface is insulated. Heat transfer according to the Newton law occurs through the surface $z = h$ with a medium whose temperature is T_0. Solve the following heat conduction boundary value problem

$$\frac{\partial^2\Theta}{\partial r^2}+\frac{1}{r}\frac{\partial\Theta}{\partial r}+\frac{a_z}{a_r}\frac{\partial^2\Theta}{\partial z^2}=\frac{1}{a_r}\frac{\partial\Theta}{\partial\tau}; \tag{1}$$

$$\Theta(r, z, 0)=0; \tag{2}$$

$$\left.\frac{\partial\Theta(r, z, \tau)}{\partial z}\right|_{z=h}=-\frac{\alpha}{\lambda_z}\Theta(r, h, \tau); \tag{3}$$

$$\left.\frac{\partial\Theta(r, z, \tau)}{\partial z}\right|_{z=0}=-\frac{q(r, \tau)}{\lambda_z}, \quad r<R; \tag{4}$$

$$\left.\frac{\partial\Theta(r, z, \tau)}{\partial z}\right|_{z=0}=0, \quad r>R. \tag{5}$$

Let us apply the Laplace and Hankel integral transforms in τ and r, respectively, to (1)-(5). Then the general solution of the transformed equation (1) is

$$\begin{aligned}\bar{\Theta}_{\text{H}}(p, z, s)&=C_1(p, s)\operatorname{ch}\left(z\sqrt{\frac{a_r}{a_z}}\sqrt{p^2+\frac{s}{a_r}}\right)+\\&+C_2(p, s)\operatorname{sh}\left(z\sqrt{\frac{a_r}{a_z}}\sqrt{p^2+\frac{s}{a_r}}\right).\end{aligned} \tag{6}$$

The following dual integral equation results from (6) and the boundary conditions (4) and (5)

$$\sqrt{\frac{a_r}{a_z}}\int_0^\infty pJ_0(pr)\sqrt{p^2+\frac{s}{a_r}}C_2(p, s)\,dp=-\frac{\bar{q}(r, s)}{\lambda_z}, \quad r<R;$$

$$\sqrt{\frac{a_r}{a_z}}\int_0^\infty pJ_0(pr)\sqrt{p^2+\frac{s}{a_r}}C_2(p, s)\,dp=0, \quad r>R.$$

To solve the equations, the function $q(r, \tau)$ must be represented in terms of a Hankel integral and the value of the discontinuous integral must be used

$$\int_0^\infty J_0(pr)J_1(pR)\,dp=\begin{cases}\dfrac{1}{R}, & r<R;\\ 0, & r>R.\end{cases}$$

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We have

$$C_2(p,\ s) = -\frac{1}{\lambda_z}\sqrt{\frac{a_z}{a_r}}\left(p^2+\frac{s}{a_r}\right)^{-\frac{1}{2}}\int\limits_0^R xJ_0(px)\bar{q}(x,\ s)\,dx.$$

The constant $C_1(p, s)$ is determined in terms of $C_2(p, s)$ from (6) and the boundary condition (3). Finally, the solution of the problem (1)-(5) in Laplace image space has the form

$$\bar{\Theta}(r,\ z,\ s) = \frac{1}{b_z}\int\limits_0^\infty \frac{pJ_0(pr)}{\sqrt{a_rp^2+s}}\left[\int\limits_0^R xJ_0(px)\bar{q}(x,\ s)\,dx\right]\times \tag{7}$$

$$\times\frac{b_z\sqrt{a_rp^2+s}\,\mathrm{ch}\left[\dfrac{(h-z)}{\sqrt{a_z}}\sqrt{a_rp^2+s}\right]+\alpha\,\mathrm{sh}\left[\dfrac{(h-z)}{\sqrt{a_z}}\sqrt{a_rp^2+s}\right]}{b_z\sqrt{a_rp^2+s}\,\mathrm{sh}\left[\dfrac{h}{\sqrt{a_z}}\sqrt{a_rp^2+s}\right]+\alpha\,\mathrm{ch}\left[\dfrac{h}{\sqrt{a_z}}\sqrt{a_rp^2+s}\right]}\,dp.$$

Let us examine the case $\alpha = \infty$. Then from (7)

$$\bar{\Theta}(r,\ z,\ s) = \frac{1}{b_z}\int\limits_0^\infty \frac{pJ_0(pr)}{\sqrt{a_rp^2+s}}\,\frac{\mathrm{sh}\left[\dfrac{(h-z)}{\sqrt{a_z}}\sqrt{a_rp^2+s}\right]}{\mathrm{ch}\left[\dfrac{h}{\sqrt{a_z}}\sqrt{a_rp^2+s}\right]}\times\int\limits_0^R xJ_0(px)\bar{q}(x,\ s)\,dx\,dp,$$

or in the space of the originals

$$\Theta(r,\ z,\ \tau) = \frac{a_z}{2\lambda_z a_r h}\int\limits_0^\tau \frac{1}{\xi}\exp\left(-\frac{r^2}{4a_r\xi}\right)\Theta_1\left(\frac{h-z}{2h}\middle|\, i\pi\frac{a_z}{h^2}\xi\right)\times \tag{8}$$

$$\times\int\limits_0^R x\exp\left(-\frac{x^2}{4a_r\xi}\right)I_0\left(\frac{xr}{2a_r\xi}\right)q(x,\ \tau-\xi)\,dx\,d\xi.$$

Here $\Theta_1\left(\frac{h-z}{2h}\middle|\, i\pi\frac{a_z}{h^2}\xi\right)$ is a theta function [1]

$$\Theta_1\left(\frac{h-z}{2h}\middle|\, i\pi\frac{a_z}{h^2}\xi\right) = \frac{h}{\sqrt{\pi a_z\xi}}\sum_{n=-\infty}^{\infty}(-1)^n\exp\left[-\frac{h^2}{a_z\xi}\left(n-\frac{z}{2h}\right)^2\right]. \tag{9}$$

Using the Jacobi imaginary transformation

$$\Theta_1(\delta|\tau) = i(-i\tau)^{-\frac{1}{2}}\exp\left(-\frac{\pi i\delta^2}{\tau}\right)\Theta_1\left(\frac{\delta}{\tau}\middle|-\frac{1}{\tau}\right)$$

we can go from (9) to another representation for the theta function

$$\Theta_1\left(\frac{h-z}{2h}\middle|\, i\pi\frac{a_z\xi}{h^2}\right) = 2\sum_{n=0}^{\infty}(-1)^n\exp\left[-\frac{\pi^2\xi a_z}{h^2}\left(n+\frac{1}{2}\right)^2\right]\times \tag{10}$$

$$\times\sin\left[\pi(2n+1)\frac{h-z}{2h}\right].$$

The theta function representations mentioned permit the temperature distribution law to be obtained in the form of a rapidly convergent series for either large values of τ (10) or for small (9).

If the heat flux density is $q(r, \tau) = q_0 = \text{const}$, then we have from (8) and (9)

$$\frac{T(r,\ z,\ \tau)-T_0}{T_0\mathrm{Ki}_z} = \frac{1}{2\sqrt{\pi}K_a}\sum_{k=0}^{\infty}\frac{(-1)^k K_a^{-k}}{(k+1)!}\,{}_2F_1(-k,\ -1-k;\ 1;\ \bar{r}^2)\times \tag{11}$$

$$\times\sum_{m=0}^{\infty}(-1)^m\left\{\frac{1}{(2m\bar{h}+\bar{z})^{2k+1}}\Gamma\left(k+\frac{1}{2},\ \frac{(2m\bar{h}+\bar{z})^2}{4\,\mathrm{Fo}_z}\right)-\right.$$

$$\left.-\frac{1}{(2\bar{h}m+2\bar{h}-\bar{z})^{2k+1}}\Gamma\left(k+\frac{1}{2},\ \frac{(2m\bar{h}+2\bar{h}-\bar{z})^2}{4\,\mathrm{Fo}_z}\right)\right\},$$

where

$$K_a = a_r/a_z = \lambda_r/\lambda_z;\quad \bar h = h/R;\quad \bar z = z/R;\quad \bar r = r/R;$$
$$\mathrm{Ki}_z = q_0 R/(\lambda_z T_0);\quad \mathrm{Fo}_z = a_z\tau/R^2.$$

The temperature field on the r = 0 axis has the form

$$\frac{T(0,\ z,\ \tau)-T_0}{T_0\mathrm{Ki}_z} = 2\sqrt{\mathrm{Fo}_z}\sum_{n=0}^{\infty}(-1)^n\left\{\mathrm{ierfc}\left(\frac{2\bar h n+\bar z}{2\sqrt{\mathrm{Fo}_z}}\right)-\right.$$
$$-\mathrm{ierfc}\left(\frac{2\bar h n+2\bar h-\bar z}{2\sqrt{\mathrm{Fo}_z}}\right)-\mathrm{ierfc}\left(\frac{\sqrt{(2\bar h n+\bar z)^2+K_a^{-1}}}{2\sqrt{\mathrm{Fo}_z}}\right)+ \tag{12}$$
$$\left.+\,\mathrm{ierfc}\left(\frac{\sqrt{(2\bar h n+2\bar h-\bar z)^2+K_a^{-1}}}{2\sqrt{\mathrm{Fo}_z}}\right)\right\}.$$

The temperature distribution on the surface z = 0 ($r \geq 0$, $\mathrm{Fo}_r = a_r\tau/R^2 > 0$) can be written in the form

$$\frac{T(r,\ 0,\ \tau)-T_0}{T_0\mathrm{Ki}_z} = \frac{2}{\pi}K_a^{-\frac{1}{2}}\mathbf{E}(\bar r)-\frac{K_a^{-\frac{1}{2}}}{2\sqrt{\mathrm{Fo}_r}}\sum_{m=0}^{\infty}\frac{(-1)^m(4\,\mathrm{Fo}_r)^{-m}}{(m+1)!\,(2m+1)}\times$$
$$\times{}_2F_2\left(m+\frac{1}{2},\ m+1;\ 1,\ m+\frac{3}{2};\ -\frac{\bar r^2}{4\,\mathrm{Fo}_r}\right)+\frac{K_a^{-1}}{2\sqrt{\pi}\,\bar h}\sum_{m=1}^{\infty}\frac{(-1)^m}{m}\times$$
$$\times\sum_{k=0}^{\infty}\frac{(-1)^k(4m^2\bar h^2K_a)^{-k}}{(k+1)!}\,{}_2F_1(-k,\ -1-k;\ 1;\ \bar r^2)\times \tag{13}$$
$$\times\,\Gamma\left(k+\frac{1}{2},\ \frac{\bar h^2m^2K_a}{\mathrm{Fo}_r}\right),$$

where $\mathbf{E}(\bar r) = \frac{\pi}{2}\,{}_2F_1\left(-\frac{1}{2},\ \frac{1}{2};\ 1;\ \bar r^2\right)$ is the complete elliptic integral of the second kind.

The change in temperature at the center of the heating spot has the following form for r = z = 0

$$\frac{T(0,\ 0,\ \tau)-T_0}{T_0\mathrm{Ki}_z} = \frac{2\sqrt{\mathrm{Fo}_z}}{\sqrt{\pi}}-2\sqrt{\mathrm{Fo}_z}\left\{\mathrm{ierfc}\left(\frac{K_a^{-\frac{1}{2}}}{2\sqrt{\mathrm{Fo}_z}}\right)-\right. \tag{14}$$
$$\left.-2\sum_{n=1}^{\infty}(-1)^n\left[\mathrm{ierfc}\left(\frac{n\bar h}{\sqrt{\mathrm{Fo}_z}}\right)-\mathrm{ierfc}\left(\frac{\sqrt{4\bar h^2n^2+K_a^{-1}}}{2\sqrt{\mathrm{Fo}_z}}\right)\right]\right\}.$$

As $h \to \infty$ solutions for iso- and orthotropic half-spaces follow from (11)-(14) [2, 4-11].

In the case of a stationary thermal mode ($\tau \to \infty$) we have from (8)

$$\frac{T(r,\ z,\ \infty)-T_0}{T_0\mathrm{Ki}_z} = K_a^{-\frac{1}{2}}\int_0^{\infty}\frac{J_0(x\bar r)J_1(x)}{x}\,\frac{\mathrm{sh}[x(\bar h-\bar z)\sqrt{K_a}]}{\mathrm{ch}[x\bar h\sqrt{K_a}]}\,dx. \tag{15}$$

For the central point r = z = 0

$$\frac{T(0,\ 0,\ \infty)-T_0}{T_0\mathrm{Ki}_z} = K_a^{-\frac{1}{2}}\int_0^{\infty}\frac{J_1(x)}{x}\,\mathrm{th}(x\bar h\sqrt{K_a})\,dx. \tag{16}$$

We use the known representation [3]

$$\mathrm{th}(x\bar h\sqrt{K_a}) = \frac{8}{\pi^2}x\bar h\sqrt{K_a}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2+\dfrac{4x^2\bar h^2K_a}{\pi^2}};$$

$$\int_0^{\infty}\frac{J_1(cx)\,dx}{x^2+\mu^2} = \frac{1}{c\mu^2}-\frac{1}{\mu}K_1(c\mu).$$

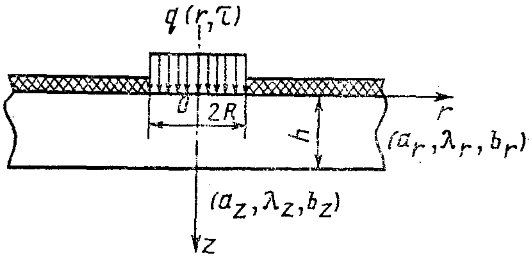


Fig. 1. Physical model of an unbounded orthotropic plate heated through a circular domain, z = 0, 0 ≤ r <.

to evaluate the integral in the right side of (16).

Finally (16) takes for form

$$\frac{T(0,\ 0,\ \infty)-T_0}{T_0\mathrm{Ki}_z}=\bar{h}-\frac{4}{\pi\sqrt{K_a}}\sum_{n=0}^{\infty}\frac{K_1\left[\frac{\pi(2n+1)}{2\bar{h}\sqrt{K_a}}\right]}{2n+1}. \tag{17}$$

If the integral in the right side of (15) is evaluated in an analogous manner, the following representations must here be utilized

$$\frac{\mathrm{sh}\,[x(\bar{h}-\bar{z})K_a^{\frac{1}{2}}]}{\mathrm{ch}\,(x\bar{h}K_a^{1/2})}=\frac{8}{\pi^2}x\bar{h}K_a^{\frac{1}{2}}\sum_{n=0}^{\infty}\frac{(-1)^n\sin\left[\frac{\pi(2n+1)(\bar{h}-\bar{z})}{2\bar{h}}\right]}{(2n+1)^2+\frac{4}{\pi^2}x^2\bar{h}^2K_a};$$

$$\int_0^{\infty}\frac{J_1(bx)\,J_0(cx)\,dx}{x^2+z^2}=\begin{cases}\frac{1}{z^2b}-\frac{1}{z}K_1(bz)\,I_0(cz), & 0<c<b;\\ \frac{1}{z}K_0(cz)\,I_1(bz), & 0<b<c.\end{cases}$$

The stationary temperature field T(r, z, ∞) has the following form for the appropriate ranges $\bar{r}<1$ and $\bar{r}>1$

$$\frac{T(r,\ z,\ \infty)-T_0}{T_0\mathrm{Ki}_z}=\bar{h}-\bar{z}-\frac{4}{\pi\sqrt{K_a}}\sum_{n=0}^{\infty}(-1)^n\sin\left[\frac{\pi(\bar{h}-\bar{z})(2n+1)}{2\bar{h}}\right]\times$$
$$\times\frac{1}{2n+1}K_1\left[\frac{\pi(2n+1)}{2\bar{h}\sqrt{K_a}}\right]I_0\left[\frac{\pi(2n+1)}{2\bar{h}\sqrt{K_a}}\bar{r}\right],\quad \bar{r}<1; \tag{18}$$

$$\frac{T(r,\ z,\ \infty)-T_0}{T_0\mathrm{Ki}_z}=\frac{4}{\pi\sqrt{K_a}}\sum_{n=0}^{\infty}(-1)^n\sin\left[\frac{\pi(\bar{h}-\bar{z})(2n+1)}{2\bar{h}}\right]\times$$
$$\times\frac{1}{2n+1}K_0\left[\frac{\pi(2n+1)}{2\bar{h}\sqrt{K_a}}\bar{r}\right]I_1\left[\frac{\pi(2n+1)}{2\bar{h}\sqrt{K_a}}\right],\quad \bar{r}>1. \tag{19}$$

Appropriate solutions to determine the temperature fields in an unbounded isotropic plate heated by an analogous source result from the dependences represented above in the particular case for $K_a = 1$. Numerical data on the formation of the dimensionless temperature field (14) at the central point (r = z = 0) of a circular heat source of constant intensity in ideal thermal contact with a part of a surface (z = 0) of an unbounded iso- and orthotropic plate are presented in Table 1 as a function of the numbers $Fo_z = a_z\tau/R^2$, $\bar{h} = h/R$ and $K_a = a_r/a_z = \lambda_r/\lambda_z = 0.1;\ 1;\ 10$. Represented for comparison in separate columns ($\bar{h} = \infty$) are appropriate values of the temperature field (14) for an iso- and orthotropic half-space heated through a circular domain by an analogous source.

The established regularities of two-dimensional nonstationary and stationary temperature field development in an unbounded iso- and orthotropic plate heated by a constant intensity circular heat source permit working out a number of new methods to determine the thermophysical characteristics (a_r, a_z, λ_r, λ_z, b_r, b_z) of iso- and orthotropic materials on specimens in the shape of plates if the theoretically postulated boundary conditions are realized in the thermophysical experiment. Thus, for instance, in the initial heating stage

TABLE 1. Values of the Relative Dimensionless Temperature $\frac{T(0,\ 0,\ \tau) - T_0}{q_0 R/\lambda_z} = f(\mathrm{Fo}_z,\ \bar{h},\ K_a)$ for an Unbounded Iso- and Orthotropic Plate at the Center of a Circular Heating Spot (14)

Fo_z	K_a = 0,1				K_a = 1,0				K_a = 10			
	$\bar{h}$ = 0,05	0,1	1,0	∞	$\bar{h}$ = 0,05	0,1	1,0	∞	$\bar{h}$ = 0,05	0,1	1,0	∞
0,001	0,03489	0,03568	0,03568	0,03568	0,03489	0,03568	0,03568	0,03568	0,03489	0,03568	0,03568	0,03568
0,003	0,04790	0,06132	0,06180	0,06180	0,04790	0,06132	0,06180	0,06180	0,04790	0,06132	0,06180	0,06180
0,005	0,04970	0,07639	0,07978	0,07978	0,04970	0,07639	0,07978	0,07978	0,04970	0,07636	0,07974	0,07974
0,007	0,04995	0,08558	0,09440	0,09440	0,04995	0,08558	0,09440	0,09440	0,04995	0,08541	0,09413	0,09413
0,009	0,04999	0,09120	0,10704	0,10704	0,04999	0,09120	0,10704	0,10704	0,04998	0,09078	0,10621	0,10621
0,011	0,04999	0,09462	0,11834	0,11834	0,04999	0,09462	0,11834	0,11834	0,04999	0,09393	0,11658	0,11658
0,015	0,04999	0,09799	0,13819	0,13819	0,04999	0,09799	0,13819	0,13819	0,04999	0,09683	0,13356	0,13356
0,017	0,05000	0,09877	0,14712	0,14712	0,05000	0,09877	0,14712	0,14712	0,04999	0,09745	0,14062	0,14062
0,019	0,05000	0,09925	0,15553	0,15553	0,05000	0,09925	0,15553	0,15553	0,04999	0,09781	0,14693	0,14693
0,021	0,05000	0,09954	0,16351	0,16351	0,05000	0,09954	0,16351	0,16351	0,04999	0,09802	0,15263	0,15263
0,040	0,05000	0,09999	0,22567	0,22567	0,05000	0,09999	0,22564	0,22564	0,04999	0,09830	0,18822	0,18822
0,060	0,05000	0,09999	0,27639	0,27639	0,05000	0,09999	0,27600	0,27600	0,04999	0,09830	0,20844	0,20844
0,080	0,05000	0,10000	0,31915	0,31915	0,05000	0,10000	0,31755	0,31755	0,04999	0,09831	0,22137	0,22137
0,100	0,05000	0,10000	0,35682	0,35682	0,05000	0,10000	0,35288	0,35288	0,05000	0,09831	0,23056	0,23056
0,200	0,05000	0,10000	0,50462	0,50462	0,05000	0,10000	0,47348	0,47389	0,05000	0,09831	0,25435	0,25443
0,400	0,05000	0,10000	0,69780	0,71355	0,05000	0,10000	0,58661	0,59521	0,05000	0,09831	0,27086	0,27208
0,600	0,05000	0,10000	0,81467	0,87279	0,05000	0,10000	0,63387	0,65914	0,05000	0,09831	0,27677	0,28006
0,800	0,05000	0,10000	0,88454	1,00418	0,05000	0,10000	0,65576	0,70006	0,05000	0,09831	0,27934	0,28485
1,000	0,05000	0,10000	0,92572	1,11591	0,05000	0,10000	0,66651	0,72910	0,05000	0,09831	0,28056	0,28813
2,000	0,05000	0,10000	0,97931	1,49858	0,05000	0,10000	0,67773	0,80458	0,05000	0,09831	0,28178	0,29632
4,000	0,05000	0,10000	0,98308	1,88222	0,05000	0,10000	0,67831	0,86040	0,05000	0,09831	0,28184	0,30213
10,000	0,05000	0,10000	0,98311	2,30560	0,05000	0,10000	0,67831	0,91116	0,05000	0,09831	0,28184	0,30731
40,000	0,05000	0,10000	0,98311	2,72083	0,05000	0,10000	0,67831	0,95544	0,05000	0,09831	0,28184	0,31176
100,000	0,05000	0,10000	0,98311	2,88135	0,05000	0,10000	0,67831	0,97180	0,05000	0,09831	0,28184	0,31340
400,000	0,05000	0,10000	0,98311	3,02137	0,05000	0,10000	0,67831	0,98590	0,05000	0,09831	0,28184	0,31481
1000,000	0,05000	0,10000	0,98311	3,07310	0,05000	0,10000	0,67831	0,99108	0,05000	0,09831	0,28184	0,31533
4000,000	0,05000	0,10000	0,98311	3,11767	0,05000	0,10000	0,67831	0,99554	0,05000	0,09831	0,28184	0,31578
10000,000	0,05000	0,10000	0,98311	3,13406	0,05000	0,10000	0,67831	0,99718	0,05000	0,09831	0,28184	0,31594
100000,000	0,05000	0,10000	0,98311	3,15335	0,05000	0,10000	0,67831	0,99911	0,05000	0,09831	0,28184	0,31613
∞	0,05000	0,10000	0,98311	3,16228	0,05000	0,10000	0,67831	1,00000	0,05000	0,09831	0,28184	0,31623

($\tau \to 0$), the value of the excess temperature at the central point ($r = z = 0$) of the circular heating spot is, according to (14), expressed in the classical form

$$\frac{T(0,\ 0,\ \tau)-T_0}{T_0 \mathrm{Ki}_z}=\frac{2\sqrt{\mathrm{Fo}_z}}{\sqrt{\pi}}, \tag{20}$$

from which the thermal activity b_z can be computed from the formula

$$b_z = 2\pi^{-1/2} q_0 \sqrt{\tau}/[T(0,\ 0,\ \tau)-T_0]. \tag{21}$$

In the stationary thermal regime $[T(r,\ 0,\ \infty) - T_0]/[T(0,\ 0,\ \infty) - T_0] = N_1 = f_1(K_a,\ \bar{h})$ or $[T(0,\ z,\ \infty) - T_0]/[T(0,\ 0,\ \infty) - N_2 = f_2(K_a,\ \bar{h})$.

Graphs of the depeendences

$$N_1=\frac{1-\dfrac{4}{\pi\bar{h}\sqrt{K_a}}\displaystyle\sum_{n=0}^{\infty}\frac{1}{2n+1}K_1\left[\frac{\pi(2n+1)}{2\bar{h}\sqrt{K_a}}\right]I_0\left[\frac{\pi(2n+1)}{2\bar{h}\sqrt{K_a}}\bar{r}\right]}{1-\dfrac{1}{\pi\bar{h}\sqrt{K_a}}\displaystyle\sum_{n=0}^{\infty}\frac{1}{2n+1}K_1\left[\frac{\pi(2n+1)}{2\bar{h}\sqrt{K_a}}\right]}, \tag{22}$$

$$\bar{r}<1,$$

and

$$N_2=\frac{1-\dfrac{z}{h}-\dfrac{4}{\pi\bar{h}\sqrt{K_a}}\displaystyle\sum_{n=0}^{\infty}(-1)^n\sin\left[\frac{\pi}{2}\left(1-\frac{z}{h}\right)(2n+1)\right]\frac{K_1\left[\dfrac{\pi(2n+1)}{2\bar{h}\sqrt{K_a}}\right]}{2n+1}}{1-\dfrac{4}{\pi\bar{h}\sqrt{K_a}}\displaystyle\sum_{n=0}^{\infty}\frac{1}{2n+1}K_1\left[\frac{\pi(2n+1)}{2\bar{h}\sqrt{K_a}}\right]}, \tag{23}$$

$$z>0,$$

are presented in Fig. 2. We find the parameter $K_a = a_r/a_z$ from ratios N_1 and N_2 known from experiment for given $\bar{h} = j/R$, $\bar{r} = r/R$ and z/h by using similar graphs or analytic-machine methods. The heat conduction coefficient λ_z (for the K_a parameter found) is determined from the formula

$$\lambda_z=\frac{q_0 h}{T(0,\ 0,\ \infty)-T_0}A(\bar{h},\ K_a), \tag{24}$$

$$A(\bar{h},\ K_a)=1-\frac{4}{\pi\bar{h}\sqrt{K_a}}\sum_{n=0}^{\infty}\frac{K_1\left[\dfrac{\pi(2n+1)}{2\bar{h}\sqrt{K_a}}\right]}{2n+1}. \tag{25}$$

The dependence $A(\bar{h},\ K_a)$ is presented in Fig. 3. We calculate the thermal diffusivity coefficient a_z from the formula

$$a_z=\lambda_z^2/b_z^2. \tag{26}$$

We find the thermal diffusivity a_r and the heat conduction coefficients λ_r from the formulas

$$a_r=a_z K_a;\quad \lambda_r=\lambda_z K_a. \tag{27}$$

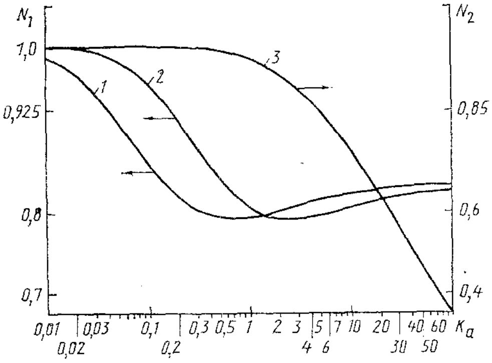
The volume specific heat $c\gamma$ is found from the relating equation $a_c c\gamma = \lambda_i$.

Using (7) for $\alpha = 0$, an expression is easily written to determine the two-dimensional nonstationary temperature field $\Theta(r,\ z,\ \tau)$ in an unbounded orthotropic plate of height (thickness) $2h$, both of whose surfaces ($z = 0$ and $z = 2h$) are heated (symmetrically) through identical circular domains $0 = r < R$ by specific thermal fluxes $q(r,\ \tau)$:

$$\begin{aligned}\Theta(r,\ z,\ \tau)=\frac{a_z}{2\lambda_z a_r h}\int_0^{\tau}\frac{1}{\xi}\exp\left(-\frac{r^2}{4a_r\xi}\right)\Theta_0\left(\frac{h-z}{2h}\,\middle|\,i\pi\frac{a_z}{h^2}\xi\right)\times\\ \times\int_0^R x\exp\left(-\frac{x^2}{4a_r\xi}\right)I_0\left(\frac{xr}{2a_r\xi}\right)q(x,\ \tau-\xi)\,dxd\xi,\end{aligned} \tag{28}$$

where

$$\Theta_0\left(\frac{h-z}{2h}\,\middle|\,i\pi\frac{a_z}{h^2}\xi\right)=\frac{h}{\sqrt{\pi a_z\xi}}\sum_{m=-\infty}^{\infty}\exp\left[-\frac{h^2}{a_z\xi}\left(m-\frac{z}{2h}\right)^2\right].$$



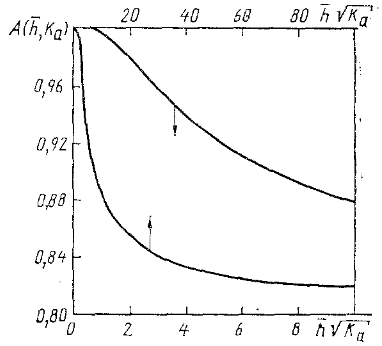


Fig. 2 Fig. 3

Fig. 2. The dependence of (22) and (23) on the parameter K_a for definite values of $\bar{h}$, $\bar{r}$, z/h: 1) $N_1 = f_1(K_a)$ for $\bar{h} = 1$, $\bar{r} = 0.75$, $z = 0$; 2) for $h = 0.5$, $r = 0.75$, $z = 0$; 3) $N_2 = f_2(K_a)$ for $\bar{h} = 0.5$, $r = 0$, $z/h = 0.5$.

Fig. 3. The dependence $A(\bar{h}, \bar{K_a}) = f(h\sqrt{K_a})$ (25)

Information about the analytic regularities of a change in the temperature $\Theta(r, z, \tau)$ at the middle of the plate (the point $r = 0$, $z = h$) and at the center of circular heating spots (the points $r = 0$, $z = 0$ or $r = 0$, $z = 2h$) is of significant interest for applied purposes of a thermophysical experiment. For $q(r, \tau) = q_0 = \text{const}$ we have from (28)

$$\frac{\Theta(0,\ 0,\ \tau)}{T_0 \mathrm{Ki}_z} = 2\sqrt{\mathrm{Fo}_z}\left\{\frac{1}{\sqrt{\pi}} - \mathrm{ierfc}\left(\frac{K_a^{-\frac{1}{2}}}{2\sqrt{\mathrm{Fo}_z}}\right) + 2\sum_{m=1}^{\infty}\left[\mathrm{ierfc}\left(\frac{m\bar{h}}{\sqrt{\mathrm{Fo}_z}}\right) - \mathrm{ierfc}\left(\sqrt{\frac{1}{4K_a\mathrm{Fo}_z} + \frac{m^2\bar{h}^2}{\mathrm{Fo}_z}}\right)\right]\right\}; \tag{29}$$

$$\frac{\Theta(0,\ h,\ \tau)}{T_0 \mathrm{Ki}_z} = 4\sqrt{\mathrm{Fo}_z}\sum_{m=1}^{\infty}\left[\mathrm{ierfc}\left(\frac{\bar{h}(2m-1)}{2\sqrt{\mathrm{Fo}_z}}\right) - \mathrm{ierfc}\left(\sqrt{\frac{1}{4K_a\mathrm{Fo}_z} + \frac{\bar{h}^2(2m-1)^2}{4\mathrm{Fo}_z}}\right)\right]. \tag{30}$$

If $K_a = 1$, then appropriate solutions result from (28)-(30) for an unbounded isotropic plate of height 2h heated from both sides (symmetrically) by a circular heat source of given density. For $K_a = 1$, $q(r, \tau) = q_0$ and $R \to \infty$ (the passage to the one-dimensional case), we arrive at appropriate solutions to determine the temperature fields $\Theta(z, \tau) = \lim\limits_{R\to\infty} \Theta(r, z, \tau)$ in an unbounded isotropic plate of height 2h, heated from both sides by constant thermal fluxes [12] from (28)-(30).

The temperature drop $\Delta T_{0,h}(\tau) = [T(0, 0, \tau) - T(0, h, \tau)]/T_0\mathrm{Ki}_z$ between the central point of the heating spot (on one of the plate surfaces) and the middle of the plate on the axis of symmetry $r = 0$ will be

$$\Delta T_{0,h}(\tau) = \frac{T(0,\ 0,\ \tau) - T(0,\ h,\ \tau)}{T_0\mathrm{Ki}_z} = 2\sqrt{\mathrm{Fo}_z}\left\{\frac{1}{\sqrt{\pi}} - \mathrm{ierfc}\left(\frac{K_a^{-\frac{1}{2}}}{2\sqrt{\mathrm{Fo}_z}}\right) + 2\sum_{m=1}^{\infty}\left[\mathrm{ierfc}\left(\frac{m\bar{h}}{\sqrt{\mathrm{Fo}_z}}\right) - \mathrm{ierfc}\left(\sqrt{\frac{1}{4K_a\mathrm{Fo}_z} + \frac{m^2\bar{h}^2}{\mathrm{Fo}_z}}\right) - \mathrm{ierfc}\left(\frac{\bar{h}(2m-1)}{2\sqrt{\mathrm{Fo}_z}}\right) + \mathrm{ierfc}\left(\sqrt{\frac{1}{4K_a\mathrm{Fo}_z} + \frac{\bar{h}^2(2m-1)^2}{4\mathrm{Fo}_z}}\right)\right]\right\}. \tag{31}$$

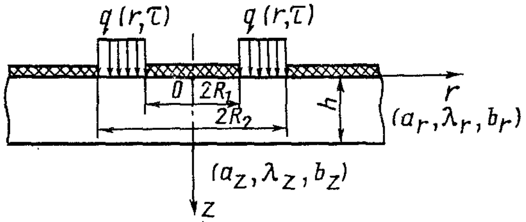


Fig. 4. Physical model of an unbounded orthotro orthotropic plate heated through an annular domain $z = 0$, $R_1 < r < R_2$.

The dependences (29)-(31) obtained can be utilized to determine the thermophysical properties of iso- and orthotropic materials in a quasistationary thermal regime.

2. HEATING THROUGH AN ANANULAR DOMAIN

Formulation of this nonstationary heat conduction problem is analogous to the preceding, except that the surface ($z = 0$) of the unbounded orthotropic plate (Fig. 4) is heated by a heat flux of density $q(r, \tau)$ acting in an annular domain $R_1 < r < R_2$. The solution of the problem in the Laplace transform space has the form

$$\bar{\Theta}(r, z, s) = \frac{1}{b_z} \int_0^\infty \frac{pJ_0(pr)}{\sqrt{a_r p^2 + s}} \left[\int_{R_1}^{R_2} x J_0(px) \bar{q}(x, s)\, dx \right] \times$$

$$\times \frac{b_z \sqrt{a_r p^2 + s}\, \mathrm{ch}\left[\frac{(h-z)}{\sqrt{a_z}} \sqrt{a_r p^2 + s}\right] + \alpha\, \mathrm{sh}\left[\frac{(h-z)}{\sqrt{a_z}} \sqrt{a_r p^2 + s}\right]}{b_z \sqrt{a_r p^2 + s}\, \mathrm{sh}\left[\frac{h}{\sqrt{a_z}} \sqrt{a_r p^2 + s}\right] + \alpha\, \mathrm{ch}\left[\frac{h}{\sqrt{a_z}} \sqrt{a_r p^2 + s}\right]} dp. \tag{32}$$

For the case $\alpha = \infty$

$$\bar{\Theta}(r, z, s) = \frac{1}{b_z} \int_0^\infty \frac{pJ_0(pr)}{\sqrt{a_r p^2 + s}} \frac{\mathrm{sh}\left[\frac{(h-z)}{\sqrt{a_z}} \sqrt{a_r p^2 + s}\right]}{\mathrm{ch}\left[\frac{h}{\sqrt{a_z}} \sqrt{a_r p^2 + s}\right]} \times$$

$$\times \int_{R_1}^{R_2} x J_0(px)\, \bar{q}(x, s)\, dx dp, \tag{33}$$

and if the specific heat flux is a function of just the time $\bar{q}(r, s) = \bar{q}(s)$ then

$$\bar{\Theta}(r, z, s) = \frac{\bar{q}(s)}{b_z} \int_0^\infty \frac{J_0(pr)}{\sqrt{a_r p + s}} \times$$

$$\times [R_2 J_1(pR_2) - R_1 J_1(pR_1)] \frac{\mathrm{sh}\left[\frac{(h-z)}{\sqrt{a_z}} \sqrt{a_r p^2 + s}\right]}{\mathrm{ch}\left[\frac{h}{\sqrt{a_z}} \sqrt{a_r p^2 + s}\right]} dp \tag{34}$$

It is not difficult to go from the transforms over to the originals in the solutions (32)-(34) since the solution (34) in integral form becomes

$$\Theta(r, z, \tau) = \frac{a_z}{2\lambda_z a_r h} \int_0^\tau \frac{1}{\xi} \exp\left(-\frac{r^2}{4a_r \xi}\right) \Theta_1\left(\frac{h-z}{2h} \middle| i\pi\xi \frac{a_z}{h^2}\right) \times$$

$$\times \int_{R_1}^{R_2} x \exp\left(-\frac{x^2}{4a_r \xi}\right) I_0\left(\frac{xr}{2a_r \xi}\right) q(x, \tau - \xi)\, dx d\xi. \tag{35}$$

The solution (35) can be written in terms of infinite series analogous to the series (9)-(14) in the case of a known function of the heat flux density $q(x, \tau)$. In general, it is easy to note that the general form of the solutions to determine the spatial temperature fields in the orthotropic unbounded plate under consideration that is heated through an annular domain, differs from the corresponding solutions in the case of local heating of this plate through a circle of known radius by the presence (in the general solution) of the definite integral $\int_{R_1}^{R_2} xJ_0(px)\bar{q}(x, s)dx$ in place of $\int_0^R xJ_0(px)\bar{q}(x, s)dx$ for heating through a circular domain. Therefore, execution of appropriate analytical investigations and computations in the case of local heating of the plate under consideration through an annular domain is of no special difficulty since the investigations performed at the beginning of the paper for heating this plate through a circular domain can be utilized.

The solutions represented for the nonstationary and stationary heat conduction problems for an unbounded iso- and orthotropic plate with given discontinuous boundary conditions permit investigation of an important methodological question of the theory and practice of thermophysical measurements associated with the optimal selection of the linear dimensions of the specimen to be tested $\bar{h} = h/R$ for which a simpler physicomathematical model of an iso- and orthotropic half-space ($h \to \infty$) heated by local heat sources [2-11] can be utilized successfully with a given degree of accuracy (in the whole time interval).

Starting from (16), we have as $h \to \infty$

$$\lim_{h\to\infty} \frac{T(0,\ 0,\ \infty)-T_0}{T_0 \mathrm{Ki}_z} = \frac{T^*(0,\ 0,\ \infty)-T_0}{T_0 \mathrm{Ki}_z} = \frac{1}{\sqrt{K_a}}, \tag{36}$$

where $T^*(0, 0, \infty)$ is the value of the stationary temperature at the center of a circular heat source of constant intensity for local heating of an orthotropic half-space [10].

Therefore, for local heating of an orthotropic material through a circular domain by a constant heat flux, the relative methodological error δ (%) in the temperature readings $[T(o, o, \infty) - T_0]$ on an orthotropic specimen of height h as compared with the temperature readings $[T^*(0, 0, \infty) - T_0$, when $h \to \infty$ will equal

$$\delta = \frac{[T^*(0,\ 0,\ \infty)-T_0]-[T(0,\ 0,\ \infty)-T_0]}{T^*(0,\ 0,\ \infty)-T_0}\cdot 100\ \% = \left\{1-\int_0^\infty \frac{J_1(x)}{x}\,\mathrm{th}(x\bar{h}\sqrt{K_a})\,dx\right\}\cdot 100\ \%. \tag{37}$$

The dependence (37) shows that satisfaction of the condition of half-boundedness ($h = \infty$) during investigations of orthotropic materials heated through a circular domain depends on the complex group $\bar{h}\sqrt{K_a}$. For an analogous investigation of isotropic materials ($K_a = 1$) the corresponding methodological estimates δ (%) will be functions of just the parameter $\bar{h} = h/R$.

The optimal selection of the diameter $2r_{spe}$ for which the physicomathematical model of an iso- and orthotropic half-space can be used with a given degree of accuracy [2-11] is found by starting from the solution of the following two-dimensional problem of stationary heat conduction for an orthotropic semi-bounded body with mixed discontinuous boundary conditions: let the semi-boundary orthotropic body ($z = 0$) be heated through a circular domain $0 \le r < r$, $z = 0$ by a constant heat flux of density q_0. Outside the limits of the heating spot $r > R$, $z = 0$ on the whole extent of the heat transfer ($\tau \to \infty$) a constant temperature is maintained equal to the initial value T_0, i.e., $T(r, 0, \infty) = T_0$ for $r > R$, $z = 0$, $\tau = \infty$. The general solution to determine the two-dimensional stationary temperature field $\Theta_{st}(r, z) = T_{st}(r, z) - T_0$ at any point r, z of an orthotropic half-space under the formulated boundary conditions can be written in the form

$$\Theta_{st}(r,\ z) = \frac{2q_0}{\pi\lambda_z\sqrt{K_a}}\int_0^\infty \exp(-pz\sqrt{K_a})\,J_0(pr)\times\{\sin pR - pR\cos pR\}\,\frac{dp}{p^2}. \tag{38}$$

For $z = 0$ we will have the value of the stationary temperature on the heating surface of an orthotropic semi-bounded body through a circle of radius $r = R$

$$\Theta_{st}(r,\ 0) = \begin{cases} \dfrac{2q_0 R}{\pi\lambda_z\sqrt{K_a}}\sqrt{1-\dfrac{r^2}{R^2}}, & r<R; \\ 0, & r>R. \end{cases} \tag{39}$$

Using (38), the ratio between the heat flux density $\lambda_z(\partial\Theta_{st}(r, 0))/\partial z$ at any point of the boundary surface $z = 0$, $r \geq 0$ and the given heat flux density q_0 in the circle domain $0 \leq r < R$, $z = 0$ can be found easily

$$\left.\frac{q^*(r)}{q_0}\right|_{z=0} = \begin{cases} -1, & r<R; \\ \dfrac{2}{\pi}\left[\dfrac{1}{\sqrt{\dfrac{r^2}{R^2}-1}} - \arcsin\left(\dfrac{R}{r}\right)\right], & r>R. \end{cases} \quad (40)$$

The expression (40) obtained is the initial equation to determine the specimen radius $r = r_{spe}$ for which the ratio $q^*(r)/q_0$ would satisfy the requisite accuracy. Thus, for example, for $r/R = r_{spe}/R = 4.8$, the ratio is $q^*(r)/g_0 = 0.001$, which is 0.1%. Therefore, the error in determining the heat conductivity $r_{spe} = 4.8R$ (for the realization of local specimen heating through a circle of radius R in the experiment) does not exceed 0.1% when a specimen radius $r_{spe} = 4.8R$ is selected, for other conditions of the experiment being ideal. It should be noted that the parameter of the material being investigated $K_a = a_r/a_z = \lambda_r/\lambda_z$. does not exert influence on the optimal selection of the specimen size from (40).

NOTATION

$\Theta = \Theta(r, z, \tau) = T(r, z, \tau) - T_0$, excess temperature at any point of an unbounded orthotropic plate; T_0, initial temperature of the plate; r, z, running cylindrical coordinates; τ, time; h, plate height (thickness); $\bar{h} = h/R$, relative (dimensionless) plate parameter characterizing its thickness; R, radius of the circular heat source; R_1, R_2, inner and outer radii of the annular heat source, respectively; α, heat transfer coefficient on the surface $z = h$ of the plate under consideration; $q(r, \tau)$, heat flux density at $z = 0$ in the circular $0 \leq r \leq R$ or annular $R_1 \leq r \leq R_2$ heating domains; λ_z, a_z, λ_r, a_r, heat conductivity and thermal diffusivity of a plate in the directions of the z and r coordinates respectively; z and r; $b_i = \lambda_i/\sqrt{a_i}$, thermal activity in the i direction; $\bar{\Theta}_H(p,z,s) = \int_0^\infty\int_0^\infty \exp(-s\tau) \times \Theta(r, z, \tau) J_0(pr) r dr d\tau$ Laplace and Hankel transform of the desired excess temperature function $\Theta(r, z, \tau)$; s,p, Laplace and Hankel integral transform parameters respectively; $J_0(x)$, $J_1(x)$, Bessel functions of real argument of zero and first orders respectively; $C_1(p, s)$, $C_2(p, s)$, constants of integration of the transformed heat conduction equation; ch(x), sh(x), hyperbolic functions; $I_\nu(x)$, $K_\nu(x)$, modified Bessel functions; $\Theta_1(v|t)$, $\Theta_0(v|t)$, theta function (according to the text); $Fo_z = a_z\tau/R^2$ and $Fo_r = a_r\tau/R^2$, Fourier numbers; $Ki_z = q_0R/(\lambda_zT_0)$ and $Ki_r = q_0R/(\lambda_rT_0)$, Kirpichev criteria; $Lv_i = q_0\sqrt{\tau}/(b_iT_0)$, Lykov criterion; $K_a = a_r/a_i = K_\lambda = \lambda_r/\lambda_z$, parameter characterizing the ratio between the thermal diffusivities and heat conductivities in appropriate directions of the relative cylindrical coordinates $\bar{z} = z/R$ and $\bar{r} = r/R$; ierfc(x), multiple probability integral; $\Gamma(\alpha, x)$, additional incomplete Gamma function; ${}_2F_1(a, b; c; x)$, Gauss hypergeometric function; $E(\bar{r}) = E(r/R)$, complete elliptical integral of the second kind; $q_0 = W_0/(\pi R^2) = \text{const}$, constant heat flux density for a circular heat (W_0 is the heat source intensity in W); $N_1 = [T(r, 0, \infty) - T_0]/[T(0, 0, \tau) - T_0]$, $N_2 = [T(0, z, \infty) - T_0]/[T(0, 0, \infty) - T_0]$, ratio of the excess temperatures in the stationary regime for corresponding points of an orthotropic plate; $\Delta T_{0,h}(\tau) = [T(0, 0, \tau) - T(0, h, \tau)]/(T_0Ki_z)$, dimensionless temperature drop over the plate section between the points $r = z = 0$ and $r = 0$, $z = h$ for the case $\alpha = 0$.

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TEMPERATURE FIELD IN A HALFSPACE WITH A PARALLELEPIPED-SHAPED HEAT-RELEASING INCLUSION

Yu. M. Kolyano, Yu. M. Krichevets, E. G. Ivanik, and V. I. Gavrysh

UDC 536.24

A study is made of the stationary temperature field in a half-space containing a foreign heat-releasing inclusion of parallelepiped shape of small dimensions.

In the operation of metalloceramic bodies of radio-electronic apparatus a need arises for studying temperature fields for bodies with foreign inclusions of small dimensions.

In this connection we consider an isotropic halfspace containing, at a distance ℓ from its boundary surface, a foreign inclusion of parallelepiped shape and volume $V_0 = 8hbd$ in whose vicinity uniformly distributed internal heat sources of strength q_0 are operative. We refer the body in question to a rectangular cartesian coordinate system. We place the coordinate origin at the center of the inclusion. On the boundary surface $z = -d-\ell$ a convective heat exchange is specified with external mean temperature t_c.

For the determination of the stationary temperature field we have the heat conduction equation [1]

$$\frac{\partial}{\partial x}\left[\lambda(x,\ y,\ z)\ \frac{\partial\Theta}{\partial x}\right]+\frac{\partial}{\partial y}\left[\lambda(x,\ y,\ z)\ \frac{\partial\Theta}{\partial y}\right]+ \\ +\frac{\partial}{\partial z}\left[\lambda(x,\ y,\ z)\ \frac{\partial\Theta}{\partial z}\right]=-Q(x,\ y,\ z), \tag{1}$$

where

$$\lambda(x,\ y,\ z)=\lambda_1+(\lambda_0-\lambda_1)N(x,\ h)N(y,\ b)N(z,\ d); \\ Q(x,\ y,\ z)=q_0N(x,\ h)N(y,\ b)N(z,\ d); \\ \Theta=t-t_c;\ N(x,\ h)=S(x+h)-S(x-h). \tag{2}$$

The boundary conditions may be written in the form

$$\lambda_1\frac{\partial\Theta}{\partial z}=\alpha_z\Theta\ \text{ for } z=-d-l,\ \Theta=0 \text{ for } z\to\infty, \\ \Theta=0,\ \frac{\partial\Theta}{\partial x}=0\ \text{ for } |x|\to\infty,\ \ \Theta=0,\ \frac{\partial\Theta}{\partial y}=0 \text{ for } |y|\to\infty. \tag{3}$$

We assume that the dimensions of the foreign inclusion are small in comparison with the distance from the coupling boundary to the boundary surface. We introduce the adduced thermal conductivity $\Lambda_0 = \lambda_0V_0$ of the inclusion, the adduced power $Q_0 = q_0V_0$ of the heat sources acting in it, and in Eqs. (2) we pass to the limit, letting $h \to 0$, $b \to 0$, $d \to 0$,

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